

Secure entanglement distillation for double-server blind quantum computation

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Blind quantum computation is a new secure quantum computing protocol where a client, who does not have enough quantum technologies at her disposal, can delegate her quantum computation to a server, who has a fully-fledged quantum computer, in such a way that the server cannot learn anything about client's input, output, and program. If the client interacts with only a single server, the client has to have some minimum quantum power, such as the ability of emitting randomly-rotated single-qubit states or the ability of measuring states. If the client interacts with two servers who share Bell pairs but cannot communicate with each other, the client can be completely classical. For such a double-server scheme, two servers have to share clean Bell pairs, and therefore the entanglement distillation is necessary in a realistic noisy environment. In this paper, we show that it is possible to perform entanglement distillation in the double-server scheme without degrading the security of the blind quantum computing.

A first generation quantum computer will be implemented in a “cloud” style, since only limited number of groups, such as huge industries and governments, will be able to possess it. When a client uses such a quantum server via a remote access, it is crucial to protect client's privacy. Blind quantum computation [1–11] is a new secure quantum computing protocol which can guarantee the security of client's privacy in such a cloud quantum computing. Protocols of blind quantum computation enable a client (Alice), who does not have enough quantum technologies at her disposal, to delegate her quantum computation to a server (Bob), who has a fully-fledged quantum computer, in such a way that Alice's input, output, and program are hidden to Bob [1–11]. The original protocol of blind quantum computation was proposed by Broadbent, Fitzsimons, and Kashefi (BFK) [1]. Their protocol uses the measurement-based quantum computation on the cluster state (graph state) by Raussendorf and Briegel [12]. The BFK protocol has been recently generalized to other blind quantum computing protocols which use the measurement-based quantum computation on the Affleck-Kennedy-Lieb-Tasaki (AKLT) state [5, 13, 14], the topologically-protected measurement-based quantum computation [6, 15], the continuous-variable measurement-based quantum computation [7, 16], and the ancilla-driven model [10, 17]. In these protocols, Alice has to emit randomly-rotated few particle states. Since a creation of a single particle state (such as a single-photon state) is generally not easy, several ideas of avoiding it has been proposed. For example, a new idea which allows Alice to prepare coherent states instead of single-photon states has been proposed [4]. New protocols of blind quantum computation for Alice who does only measurements have also been proposed [9]. Since preparation of coherent states and measurements of polarizations with threshold detectors are considered to be much easier than the single-photon generation, these new ideas ease Alice's burden. Some

methods (so called “verification methods”) of checking whether Bob is doing a correct computation have also been proposed [2, 8]. Finally, a proof-of-principle experiment of the BFK protocol has been achieved recently with a quantum optical system [3].

Let us briefly review the BFK blind protocol [1]. For details see Ref. [1]. Assume that Alice wants to perform the measurement-based quantum computation on the m -qubit graph state corresponding to the graph G . The quantum algorithm which Alice wants to run is specified with the measurement basis $\{|0\rangle \pm e^{i\phi_j}|1\rangle\}$ for j th qubit ($j = 1, 2, \dots, m$), where $\phi_j \in \{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$. (Note that such $X - Y$ plain measurements are universal [12].) The BFK protocol runs as follows (see also Fig. 1):

- S1. Alice tells Bob the graph G [18].
- S2. Alice sends Bob $\bigotimes_{j=1}^m |\theta_j\rangle$, where $|\theta_j\rangle \equiv |0\rangle + e^{i\theta_j}|1\rangle$ and θ_j is randomly chosen by Alice from $\{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$.
- S3. Bob makes $|G\{\theta_j\}\rangle \equiv \left(\bigotimes_{(i,j) \in E} CZ_{i,j}\right) \bigotimes_{j=1}^m |\theta_j\rangle$, where E is the set of edges of G and $CZ_{i,j}$ is the CZ gate between i th and j th qubits.
- S4. Alice and Bob now perform the measurement-based quantum computation on $|G\{\theta_j\}\rangle$ with two-way classical communications as follows: when Alice wants Bob to measure j th qubit ($j = 1, 2, \dots, m$) of $|G\{\theta_j\}\rangle$, she sends Bob $\delta_j \equiv \theta_j + \phi'_j + r_j\pi$, where $r_j \in \{0, 1\}$ is a random binary chosen by Alice and ϕ'_j is the modified version of ϕ_j according to the previous measurement results, which is the standard feed-forwarding of the one-way model [12]. Bob measures j th qubit in the basis $\{|0\rangle \pm e^{i\delta_j}|1\rangle\}$ and tells the measurement result to Alice.

We call this protocol the single-server protocol, since there is only a single server (Bob). Because Alice's computational angle ϕ_j is “one-time padded” with a random

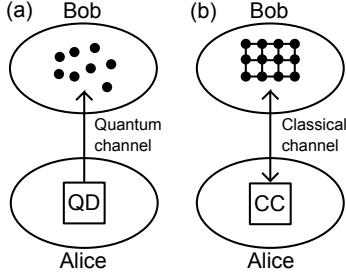


FIG. 1: The single-server blind protocol. (a) Alice sends many single-qubit states to Bob. QD is a device which emits randomly rotated single qubits. (b) Bob creates a resource state. Alice and Bob perform the measurement-based quantum computation through the two-way classical channel. CC is a classical computer.

angle θ_j , Bob cannot learn ϕ_j from δ_j (for rigorous proofs of the security of the BFK protocol, see Ref. [1, 2, 11]). Furthermore, if Bob is honest, Alice and Bob can perform the correct measurement-based quantum computation, since $|G\{\theta_j\}\rangle = \left(\bigotimes_{(i,j) \in E} CZ_{i,j}\right) \bigotimes_{j=1}^m |\theta_j\rangle = \left(\bigotimes_{j=1}^m e^{-iZ\theta_j/2}\right) \left(\bigotimes_{(i,j) \in E} CZ_{i,j}\right) |+\rangle^{\otimes m}$, where $|+\rangle \equiv |0\rangle + |1\rangle$ and $Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$, and therefore θ_j in δ_j is nicely canceled. In other words, what Bob does is effectively the measurement-based quantum computation on the graph state corresponding to the graph G with the measurement angles $\{\phi_j\}_{j=1}^m$.

In the above single-server protocol, Alice has to have the ability of emitting randomly-rotated single-qubit states, $\{|\theta_j\rangle\}_{j=1}^m$. It was shown in Ref. [1] that if we have two servers (Bob1 and Bob2) who share Bell pairs but cannot communicate with each other, Alice can be completely classical. (Alice has only to have a classical computer and two classical channels, one is between Alice and Bob1 and the other is between Alice and Bob2.) We call such a scheme the double-server scheme, since there are two servers. A protocol of the double-server scheme runs as follows [1] (see also Fig. 2):

- D1. A trusted center distributes Bell pairs to Bob1 and Bob2 [19]. Now Bob1 and Bob2 share m Bell pairs, $(|00\rangle + |11\rangle)^{\otimes m}$.
- D2. Alice sends Bob1 classical messages $\{\theta_j\}_{j=1}^m$, where θ_j is randomly chosen by Alice from $\{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$.
- D3. Bob1 measures his qubit of the j th Bell pair in the basis $\{|0\rangle \pm e^{-i\theta_j} |1\rangle\}$ ($j = 1, \dots, m$). Bob1 tells Alice the measurement results $\{b_j\}_{j=1}^m \in \{0, 1\}^m$.
- D4. After these Bob1's measurements, what Bob2 has is $\bigotimes_{j=1}^m Z_j^{b_j} |\theta_j\rangle = \bigotimes_{j=1}^m |\theta_j + b_j\pi\rangle$. Now Alice and Bob2 can start the single-server BFK protocol with the modification $\{\theta_j\}_{j=1}^m \rightarrow \{\theta_j + b_j\pi\}_{j=1}^m$.

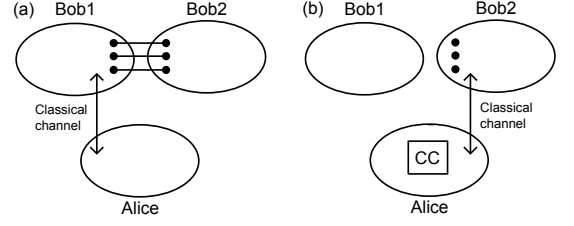


FIG. 2: The double-server blind protocol. (a) Bob1 and Bob2 share Bell pairs. Alice sends classical messages to Bob1. Bob1 performs measurements on his qubits of the Bell pairs, and tells the measurement results to Alice. (b) Alice and Bob2 run the single-server blind protocol through the two-way classical channel. CC is a classical computer.

In addition to the advantage of the completely classical Alice, the double-server scheme is intensively studied in computer science in the context of the multi-prover interacting proof system and device-independent quantum key distribution [1, 20, 21].

Note that the impossibility of the communication between two Bobs is crucial in the double-server protocol. If Bob1 can send some message to Bob2, Bob1 can tell Bob2 $\{\theta_j + b_j\pi\}_{j=1}^m$, and then Bob2 can learn something about $\{\phi_j\}_{j=1}^m$, since Bob2 knows $\{\theta_j + b_j\pi + \phi'_j + r_j\pi\}_{j=1}^m$. On the other hand, if Bob2 can tell Bob1 $\{\theta_j + b_j\pi + \phi'_j + r_j\pi\}_{j=1}^m$, Bob1 can learn something about $\{\phi_j\}_{j=1}^m$, since Bob1 knows $\{\theta_j + b_j\pi\}_{j=1}^m$. In these cases, the security of Alice's privacy is no longer guaranteed.

In order to perform the correct double-server protocol, two Bobs must share clean Bell pairs. Sharing clean Bell pairs is also crucial in many other quantum information protocols such as the quantum teleportation [22], the quantum key distribution [23, 24] and the distributed quantum computation [25–29]. One standard way of sharing clean Bell pairs in a noisy environment is the entanglement distillation [30–33]. In entanglement distillation protocols, two people, say Bob1 and Bob2, who want to share clean Bell pairs start with some dirty n Bell pairs. Then they perform local operations with some classical communications, and finally “distill” m ($m < n$) clean Bell pairs [30–33].

If we consider the application of the entanglement distillation to the double-server blind protocol, one huge problem is that two Bobs are not allowed to communicate with each other in the double-server scheme. Hence, message exchanges between two Bobs, which are necessary for the entanglement distillation, must be done through the Alice's mediation, i.e., Bob1 (Bob2) sends a message to Alice, and Alice transfers it to Bob2 (Bob1). It is not self-evident that the security of the double-server blind protocol is guaranteed even if we plug an entanglement distillation protocol into the double-server blind protocol. For example, Bob1 might send a message to Alice pretending that it is a “legal” message for the entanglement

distillation. Alice might naively forward that message to Bob2 without noticing Bob1's evil intention and believing that it is a harmless message. In this case, Bob1 can indirectly send some message to Bob2, and hence the security of the double-server protocol is no longer guaranteed.

If the entire entanglement distillation is completed before starting the double-server protocol, and if Alice delegates her computation to Bobs only once, then the communication between two Bobs during the entanglement distillation is harmless, since when they are doing the entanglement distillation, messages related to Alice's computation are not yet sent to Bobs. However, if Alice delegates more than twice, then two Bobs might exchange information about the previous double-server computation during the entanglement distillation for the next round of the computation as in the case of the "device-independence" argument of the quantum key distribution with devices having memory [34]. Furthermore, the entanglement distillation might be done in parallel with the double-server protocol in order to avoid a decoherence. In these cases, we must be careful about the communication between two Bobs during the entanglement distillation.

In this paper, we show that it is possible to perform entanglement distillation in the double-server scheme without degrading the security of blind quantum computing. Throughout this paper, we denote four Bell states by $|\psi_{z,x}\rangle \equiv (I \otimes X^x Z^z)(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$, where $(z, x) \in \{0, 1\}^2$ and $X \equiv |0\rangle\langle 1| + |1\rangle\langle 0|$.

Protocol.— As in the case of the original BFK double-server protocol, a trusted center generates n Bell states, $|\psi_{00}\rangle^{\otimes n}$, and distribute them to two Bobs; one qubit of each $|\psi_{00}\rangle$ is sent to Bob1 and the other to Bob2. Due to the noise in the channel between the center and Bobs, each Bell state decoheres, $|\psi_{00}\rangle \rightarrow \rho$. Hence two Bobs share n impure pairs $\rho^{\otimes n}$, where ρ is a dirty Bell state: one qubit of ρ is possessed by Bob1 and the other is by Bob2. Without loss of generality, we can assume that ρ is the Werner state, $\rho = F\psi_{11} + \frac{1-F}{3}(\psi_{00} + \psi_{01} + \psi_{10})$, where $\psi \equiv |\psi\rangle\langle\psi|$. If it is not the Werner state, it can be converted into the Werner state by using the twirling operation (after applying $I \otimes XZ$) [31]. In order to perform the twirling operation, Alice has only to randomly choose a $SU(2)$ operator, and tell its classical description to two Bobs. Therefore the twirling operation does not affect the security.

Since ρ is Bell-diagonal, $\rho^{\otimes n}$ is the mixture of tensor products of Bell states:

$$\rho^{\otimes n} = \sum_{(z_1, x_1, \dots, z_n, x_n) \in \{0, 1\}^{2n}} p(z_1, x_1, \dots, z_n, x_n) \bigotimes_{j=1}^n \psi_{z_j, x_j}.$$

Alice randomly chooses a $2n$ -bit string s_1 and sends it to two Bobs. This s_1 is chosen completely randomly being independent of other parameters (such as θ_j , ϕ_j , etc.). Each Bob then performs certain local unitary operation

which is determined by s_1 . (The detail of the unitary operation, which is irrelevant here, is given in Ref. [31].) Each Bob measures a qubit of a single pair in the computational basis, and tells the measurement result to Alice. (Which pair is measured is also determined by s_1 [31].) From these measurement results, Alice can gain a single bit of information about $p(z_1, x_1, \dots, z_n, x_n)$ [31].

Since a single pair is measured out, now two Bobs share $n - 1$ pairs. If Alice and two Bobs repeat a similar procedure (i.e., Alice randomly chooses a $2(n - 1)$ -bit string s_2 and tells it to two Bobs. Two Bobs perform local operations, measure a single pair in the computational basis, and tell the measurement results to Alice), Alice can gain another single bit of information about $p(z_1, x_1, \dots, z_n, x_n)$.

The probability distribution $p(z_1, x_1, \dots, z_n, x_n)$ has almost all its weight for a set of $\sim 2^{nS(\rho)}$ "likely" strings, where $S(\rho)$ is the von Neumann entropy of ρ . The probability that a $2n$ -bit string $(z_1, x_1, \dots, z_n, x_n)$ falls outside of the set of the $2^{n(S(\rho)+\epsilon)}$ most probable strings is $O(e^{-\epsilon^2 n})$ [31]. Therefore, Alice can (almost) specify $p(z_1, x_1, \dots, z_n, x_n)$ if she gains $nS(\rho)$ bits of information about $p(z_1, x_1, \dots, z_n, x_n)$. This means that it is sufficient for Alice and two Bobs to repeat the above procedure for $nS(\rho)$ times. Then, two Bobs spend $nS(\rho)$ pairs for measurements, and therefore at the end of the distillation they share $m \equiv n - nS(\rho)$ pairs, $\bigotimes_{j=1}^m |\psi_{z_j, x_j}\rangle$, where $(z_j, x_j) \in \{0, 1\}^2$. Alice knows the $2m$ -bit string $(z_1, x_1, \dots, z_m, x_m)$.

After the distillation, Alice and two Bobs can start the double-server protocol. Now we modify the double-server protocol as follows:

- D1'. Two Bobs share $\bigotimes_{j=1}^m |\psi_{z_j, x_j}\rangle$.
- D2'. Alice sends Bob1 classical messages $\{\theta'_j \equiv (-1)^{x_j} \theta_j + z_j \pi\}_{j=1}^m$, where θ_j is randomly chosen by Alice from $\{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$.
- D3'. Bob1 measures his qubit of the j th Bell pair in the basis $\{|0\rangle \pm e^{-i\theta'_j} |1\rangle\}$ ($j = 1, \dots, m$). Bob1 tells Alice the measurement results $\{b_j\}_{j=1}^m \in \{0, 1\}^m$.
- D4'. The same as D4.

Since D4' is the same as D4, it is obvious that Alice can run the correct single-server blind quantum computation with Bob2.

Bob1 cannot send any message to Bob2.— Let us show that Bob1 cannot send any message to Bob2. What Bob2 receives from Alice are bit strings, s_1, \dots, s_{n-m} , and $\{\theta_j + b_j \pi + \phi'_j + r_j \pi\}_{j=1}^m$. Since s_1, \dots, s_{n-m} are completely uncorrelated with what Bob1 sends to Alice, Bob2 cannot gain any information about Bob1's message from s_1, \dots, s_{n-m} . Since $\theta_j + b_j \pi + \phi'_j + r_j \pi$ is related with b_j , which is sent from Bob1 to Alice, one might think that Bob1 can hide his message into b_j , and send it to

Bob2 through $\theta_j + b_j\pi + \phi'_j + r_j\pi$. However, r_j is randomly taken by Alice from $\{0,1\}$ being independent of what Bob1 sends to Alice. Therefore, Bob2 cannot gain any information about b_j from $\theta_j + b_j\pi + \phi'_j + r_j\pi$. Bob1 and Bob2 share entangled pairs. However, due to the no-signaling principle, only sharing entangled pairs is useless for message transmission. Hence Bob1 cannot send any message to Bob2.

Bob2 cannot send any message to Bob1.— Next let us show that Bob2 cannot send any message to Bob1. What Bob1 receives from Alice are bit strings, s_1, \dots, s_{n-m} , and $\{\theta'_j \equiv (-1)^{x_j}\theta_j + z_j\pi\}_{j=1}^m$. Again, s_1, \dots, s_{n-m} , are useless for the message transmission from Bob2 to Bob1. Since x_j and z_j are related with Bob2's measurement results during the distillation, one might think that θ'_j can be used for the message transmission from Bob2 to Bob1. However, θ_j is randomly chosen by Alice from $\{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$ being independent of what Bob2 sends to Alice and $(z_1, x_1, \dots, z_m, x_m)$. Therefore, Bob1 cannot gain any information about Bob2's message from θ'_j . Again, sharing entangled pairs is useless. Hence Bob2 cannot send any message to Bob1 [35].

Two Bobs cannot learn Alice's computational information.— Finally, let us show the security of Alice's computational information. First, from Bob2's view point, the difference between our protocol (i.e., the distillation plus the modified double-server protocol) and the original BFK double-server protocol is only that Bob2 receives bit strings, s_1, \dots, s_{n-m} , from Alice. However, these bit strings are completely uncorrelated with Alice's computational information. Therefore, our protocol is as secure as the original BFK double-server protocol against Bob2.

Second, from Bob1's view point, the differences between our protocol and the original BFK double-server protocol are

- (i) Bob1 receives bit strings, s_1, \dots, s_{n-m} , from Alice.
- (ii) Bob1 receives $\theta'_j \equiv (-1)^{x_j}\theta_j + z_j\pi$ instead of θ_j from Alice ($j = 1, 2, \dots, m$).

Again, we can safely ignore (i), since those bit strings are uncorrelated with Alice's computational information. Regarding (ii): since θ_j is randomly taken from $\{\frac{k\pi}{4} | k = 0, 1, \dots, 7\}$ being independent of Alice's computational information and $(z_1, x_1, \dots, z_m, x_m)$, Bob1 cannot gain any information about Alice's computation from θ'_j . Since θ'_j is as random as θ_j , our protocol is as secure as the original double-server BFK protocol against Bob1 [36].

Discussion.— Let us point out that there is another interesting proof of the security: we can show that our protocol can be mapped into an instance of the BFK double-server protocol with certain deviations by two Bobs. The BFK double-server protocol is secure against any Bobs' deviations. Hence, the violation of the security of our

protocol means the violation of that of the BFK double-server protocol.

First, in our protocol, states sent from the center to two Bobs decohere. In the BFK double-server protocol, two Bobs can do any operations on the Bell pairs sent from the center, and such operations can be accidentally equivalent to the noise [37]. Second, in our protocol, the distillation is performed. However, in the BFK double-server protocol, Bob1 can generate random bit strings, s_1, \dots, s_{n-m} , by himself, and perform the local unitary operations and measurements specified by s_1, \dots, s_{n-m} . Bob2 can also generate random bit strings, s'_1, \dots, s'_{n-m} , by himself, and perform the local unitary operations and measurements specified by s'_1, \dots, s'_{n-m} . It is possible that accidentally $s_j = s'_j$ for all j [37]. Finally, in the step D2', Alice sends Bob1 $\{\theta'_j = (-1)^{x_j}\theta_j + z_j\pi\}_{j=1}^m$, and in the step D3', Bob1 does the measurements in these angles. However, in the BFK double-server protocol, Bob1 can deviate in the following way: in the step D2, although he receives $\{\theta_j\}_{j=1}^m$ from Alice, he changes it into $\{\eta_j \equiv (-1)^{\alpha_j}\theta_j + \beta_j\pi\}_{j=1}^m$, where $(\alpha_j, \beta_j) \in \{0,1\}^2$ is randomly chosen by Bob1. In the step D3, Bob1 does the measurements in the basis $\{|0\rangle \pm e^{-i\eta_j}|1\rangle\}_{j=1}^m$. It is possible that accidentally $(\alpha_j, \beta_j) = (x_j, z_j)$ for all j [37].

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 - [35] Another proof is to use the no-signaling principle. Note that Alice can send bit strings, s_1, \dots, s_{n-m} , and $\{\theta'_j \equiv (-1)^{x_j} \theta_j + z_j \pi\}_{j=1}^m$ to Bob1 *before* the center sends Bell pairs to Bobs, since Alice can first randomly choose $\{\theta'_j\}_{j=1}^m$, send them to Bob1, and later “decide” $\{\theta_j\}_{j=1}^m$ after $(z_1, x_1, \dots, z_m, x_m)$ is fixed. If Bob2 can send any message to Bob1 by doing some operation on his part of the Bell pairs, it contradicts to the no-signaling principle.
 - [36] Again, the no-signaling principle can be used to show the security against Bob1: First, Alice sends all her messages to Bob1 *before* the center sends Bell pairs to Bobs. At this moment, Alice's program does not exist even in her mind, and therefore two Bobs cannot learn her program. Second, after the Bell pairs arrive at Bobs' laboratories, Alice decides her program, and starts the protocol. Now what connect Bob1 to the outside of his laboratory are only shared Bell pairs. Due to the no-signaling principle, Bob1 cannot gain any information from Bob2.
 - [37] The probability that two Bobs accidentally do such an operation is very small. However, it was shown [1] that the BFK double-server protocol is secure against *any* Bobs' operation even if its success probability is very small. Do not confuse the success probability of the operation with the success probability that Bobs correctly guess Alice's information. The definition of the security is that whatever Bobs do, the probability distribution (certainty) of Alice's information is not changed. Even if Alice's information is completely unknown to Bobs, Bobs can randomly guess her information, and it can be accidentally correct with an exponentially small probability. However, it does not mean “Bobs learn Alice's information”.